

Analytical maximum likelihood estimation of stellar magnetic fields

M. J. Martínez González^{1,2}, R. Manso Sainz^{1,2}, A. Asensio Ramos^{1,2}, and L. Belluzzi^{1,2}

¹*Instituto de Astrofísica de Canarias, Vía Láctea s/n, 38205, La Laguna, Tenerife, Spain*

²*Departamento de Astrofísica, Universidad de La Laguna, 38205, La Laguna, Tenerife*

ABSTRACT

The polarised spectrum of stellar radiation encodes valuable information on the conditions of stellar atmospheres and the magnetic fields that permeate them. In this paper, we give explicit expressions to estimate the magnetic field vector and its associated error from the observed Stokes parameters. We study the solar case where specific intensities are observed and then the stellar case, where we receive the polarised flux. In this second case, we concentrate on the explicit expression for the case of a slow rotator with a dipolar magnetic field geometry. Moreover, we also give explicit formulae to retrieve the magnetic field vector from the LSD profiles without assuming mean values for the LSD artificial spectral line. The formulae have been obtained assuming that the spectral lines can be described in the weak field regime and using a maximum likelihood approach. The errors are recovered by means of the hermitian matrix. The bias of the estimators are analysed in depth.

Key words: techniques: polarimetric – stars: magnetic field

1 INTRODUCTION

The most precise measurements of stellar magnetic fields are based on the observation and the interpretation of polarisation in spectral lines. Most of the magnetic fields detected in stars at different phases of evolution have been found by means of the Zeeman effect, which generates linear and circular polarisation in the presence of a magnetic field (e.g., Wade et al. 2000; Bagnulo et al. 2002; Jordan, Werner & O’Toole 2005; Aznar Cuadrado et al. 2004; O’Toole et al. 2005; Silvester et al. 2009; Leone et al. 2011). Here we present analytical expressions for inferring the magnetic field vector from the observed Stokes profiles induced by the Zeeman effect in the weak field approximation.

The weak field approximation is broadly applied for the inference of solar and stellar magnetic fields from the observation of the Stokes profiles. It is an analytical solution to the radiative transfer equation whose main basic assumption is that the magnetic field is sufficiently weak throughout the whole region of the atmosphere where a spectral line is formed. Although simple, this approximation is very useful since non-magnetic mechanisms usually dominate the shape of spectral lines, in both the solar and the stellar case.

For example, in the quiet Sun, which occupies the vast majority of the solar surface (far from the sunspots that harbour very strong fields), the spectral lines are well described by the weak field approximation. This has allowed the success of many synoptic magnetographs like those

of Big Bear (Sirock et al. 2001; Varsik 1995). It is even used to produce modern vector magnetograms like those obtained with the IMaX instrument (Martínez Pillet et al. 2011) onboard the Sunrise balloon (Solanki et al. 2010). The weak field approximation has also been used to diagnose the chromosphere (Merenda et al. 2006; Asensio Ramos, Trujillo Bueno & Landi Degl’Innocenti 2008) due to the enhanced thermal width of the spectral lines.

In night-time spectro-polarimetry, the weak field approximation is at the base of the least-squares deconvolution (LSD; Donati et al. 1997), the most successful technique used to detect and measure magnetic fields in solar-type stars or other stars in which the polarisation signal per spectral line is well below the noise level. Other works have used this approximation to diagnose magnetic fields in a large variety of stellar objects. A limited selection includes some recent works on central stars of planetary nebulae (Jordan, Werner & O’Toole 2005; Leone et al. 2011), white dwarfs (Aznar Cuadrado et al. 2004), pulsating stars (Silvester et al. 2009), hot subdwarfs (O’Toole et al. 2005), Ap and Bp stars (Wade et al. 2000) and chemically peculiar stars (Bagnulo et al. 2002).

2 THE WEAK FIELD APPROXIMATION

The weak field approximation is an analytical solution to the radiative transfer equation. The fundamental assumption is

that the magnetic field vector is constant and its intensity is sufficiently weak throughout the whole region of the atmosphere where an spectral line is formed. Additionally, the line-of-sight velocity and any broadening mechanism have to be constant with height in the line formation region. As a consequence, the magnetic field can be considered as a perturbation to the zero-field case. Quantitatively, the approximation holds whenever $\Delta\lambda_B \ll \Delta\lambda_D$, where $\Delta\lambda_B$ is the Zeeman splitting and $\Delta\lambda_D$ is the dominant broadening mechanism (thermal, rotation, etc.). From its definition, it is clear that the weak field regime occurs at different field strengths for different spectral lines (depending on the sensitivity to the magnetic field, the local temperature and the atomic mass) and for different stellar objects (depending on any non-magnetic broadening mechanism for the spectral lines, the rotation being the most efficient mechanism in cool stars).

To first order in $\Delta\lambda_B$, the intensity profile of a spectral line formed in a weak magnetic field is insensitive to the magnetic field. In other words, it fulfils the transfer equation in the absence of a magnetic field. At this first order, the circular polarisation profile, i. e., the Stokes V profile, for a given spectral line has the following expression:

$$V(x) = -C\Lambda g B_{||} \frac{\partial I(x)}{\partial x}. \quad (1)$$

The symbol $B_{||}$ stands for the longitudinal component of the magnetic field, i. e., $B_{||} = B \cos \theta_B$, where θ_B is the inclination of the magnetic field with respect to the observer's line of sight ($\hat{\Omega}$; see Fig. 1) and B is the magnetic field intensity. The symbol g represents the effective Landé factor of the line, which quantifies the magnetic sensitivity of the line. The effective Landé factor only depends on the quantum numbers of the transition (see, e.g., Landi Degl'Innocenti & Landolfi 2004). Equation 1 is expressed in terms of the generic wavelength variable x . If x represents the wavelength λ , then the parameter Λ equals the central wavelength of the line, λ_0^2 . However, it is customary in stellar spectropolarimetry to consider x as the velocity in Doppler units. In such a case, we have $\Lambda = c\lambda_0$, with c the speed of light. The constant $C = 4.67 \times 10^{-13} \text{ G}^{-1} \text{ \AA}^{-1}$.

At first order in $\Delta\lambda_B$, both Stokes Q and U are zero. In order to obtain an expression for the Stokes profiles characterizing linear polarisation we have to expand the radiative transfer equation to second order in $\Delta\lambda_B$ and assume that the spectral line is not saturated. Under these assumptions, the general formulae for the Stokes Q and U are:

$$\begin{aligned} Q(x) &= -\frac{C^2}{4} \Lambda^2 G B_{\perp}^2 \cos 2\phi_B \frac{\partial^2 I(x)}{\partial x^2}, \\ U(x) &= -\frac{C^2}{4} \Lambda^2 G B_{\perp}^2 \sin 2\phi_B \frac{\partial^2 I(x)}{\partial x^2}. \end{aligned} \quad (2)$$

The symbol G plays the role of the effective Landé factor for linear polarisation and quantifies the sensitivity of linear polarisation to the magnetic field. Again, it is only a function of the quantum numbers of the transition (see, e.g., Landi Degl'Innocenti & Landolfi 2004). The symbol $B_{\perp} = B \sin \theta_B$ is the component of the magnetic field perpendicular to the line of sight. The angle ϕ_B represents the azimuth of the magnetic field vector with respect to an arbitrary reference direction (\vec{e}_a, \vec{e}_b in Fig. 1 show the coordinates chosen for the reference of $Q > 0$).

When observing the polarised light in resolved sources like the Sun, we detect specific intensities. However, when observing unresolved stars, the incoming polarised radiation is obtained as an integration in the plane of the sky of the individual Stokes parameters at each point of the stellar surface. As a consequence, the specific values of the Stokes flux vector depends on the surface distribution of the magnetic field, on the centre-to-limb variation (CLV) of the radiation, and the Doppler effect due to stellar rotation. In this paper we neglect rotation, the derived expressions being valid for those cases where the broadening mechanisms dominate over rotation. In the Sun, since we observe local profiles, the main broadening mechanism is typically thermal. In general, the polarimetric signals of 99% of the solar surface, the so-called quiet Sun, can be explained in terms of the weak field regime in most spectral lines in the visible and the near-IR. For non-resolved objects, the applicability depends strongly on the actual rotational velocity as compared with the temperature, the observed spectral line, the spectral resolution, the magnetic field organisation, etc. The thermal broadening depends on the square root of the ratio between the temperature and the atomic weight of the atom. Therefore, in general, the larger the temperature and the lightest the atom, the larger the allowed rotational velocity. For low resolution spectrographs ($R \sim 2000$), basically the only remaining spectral lines (that are not wiped out by the lack of spectral resolution) are from hydrogen. For these lines, for a temperature of $T \sim 10000 \text{ K}$, the maximum line of sight velocity is $10 - 20 \text{ km s}^{-1}$. However, when observing metal lines at spectral resolutions as high as $R \sim 60000$, the allowed velocities are one order of magnitude lower.

It is customary to introduce a parametrised form of the CLV. In this work, we assume a quadratic form (e.g., Claret 2000; Cox 2000), which gives a good balance between the quality of the CLV and the simplicity of the analytical expressions presented in this paper for the polarised fluxes:

$$I(x, \mu) = I_0(x) (1 - u - v + u\mu + v\mu^2) = I_0(x)f(\mu), \quad (3)$$

where $I_0(x)$ is the intensity profile at disc centre ($\mu = 1$). The parameters u and v have values between 0 and 1 and are supposed to be constant along the spectral line (although they can vary from line to line). Note that the values of u and v have to fulfil the condition that $I(x) > 0$. The CLV is given in terms of $\mu = \cos \Theta$, where Θ is the astrometric angle between the normal to a point in the stellar surface and the line of sight. Using this law for the CLV (and neglecting rotation) the flux for Stokes I is:

$$\mathcal{F}_I(x) = I_0(x) \int d\Sigma' f(\mu), \quad (4)$$

where the integral is computed over the visible surface in the plane of the sky Σ' . For an arbitrary function $h(\rho, \alpha)$ of the polar coordinates of the stellar surface ρ and α , we have

$$\int \int d\Sigma' h(\rho, \alpha) = \int_0^1 \rho d\rho \int_0^{2\pi} d\alpha h(\rho, \alpha). \quad (5)$$

Note that the variable $\rho = \sin \Theta$, which means that $\mu = \sqrt{1 - \rho^2}$. Plugging the quadratic expression considered for the CLV, the final closed expression for the integrated Stokes I is found to be:

$$\mathcal{F}_I(x) = \pi I_0(x) \left(\frac{6 - 2u - 3v}{6} \right). \quad (6)$$

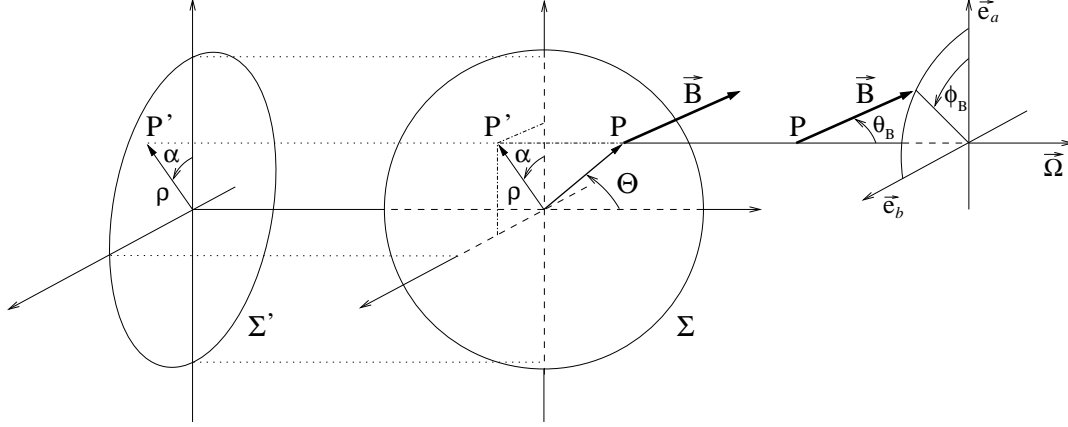


Figure 1. Geometry of the stellar model. The symbol $\vec{\Omega}$ represents the line of sight. The vector \vec{B} displays the magnetic field vector in the stellar surface. Its geometry is described in terms of the inclination with respect to the line of sight θ_B and the azimuthal angle ϕ_B (the axis \vec{e}_a being the 0 reference for the azimuth). The position of a point P in the stellar surface (Σ) is defined only by the astrocentric angle Θ . Its projection (P') on the plane of the sky (Σ') is represented in cylindrical coordinates by the modulus ρ and the angle α , which is referring to the \vec{e}_a and \vec{e}_b axis.

Integrating Eqs. (1) and (2) over the visible surface, we end up with the following expressions for the polarised flux:

$$\begin{aligned}\mathcal{F}_V(x) &= -C\Lambda g\langle B_{\parallel} \rangle \frac{\partial \mathcal{F}_I(x)}{\partial x}, \\ \mathcal{F}_Q(x) &= -\frac{C^2}{4}\Lambda^2 G\langle B_{\perp}^2 \cos 2\phi_B \rangle \frac{\partial^2 \mathcal{F}_I(x)}{\partial x^2}, \\ \mathcal{F}_U(x) &= -\frac{C^2}{4}\Lambda^2 G\langle B_{\perp}^2 \sin 2\phi_B \rangle \frac{\partial^2 \mathcal{F}_I(x)}{\partial x^2},\end{aligned}\quad (7)$$

with the definitions:

$$\begin{aligned}\langle B_{\parallel} \rangle &= \frac{6}{\pi(6-2u-3v)} \int d\Sigma' B_{\parallel} f(\mu) \\ \langle B_{\perp}^2 \cos 2\phi_B \rangle &= \frac{6}{\pi(6-2u-3v)} \int d\Sigma' B_{\perp}^2 \cos 2\phi_B f(\mu) \\ \langle B_{\perp}^2 \sin 2\phi_B \rangle &= \frac{6}{\pi(6-2u-3v)} \int d\Sigma' B_{\perp}^2 \sin 2\phi_B f(\mu).\end{aligned}\quad (8)$$

Therefore, the weak-field approximation for the integrated polarised flux remains formally the same but the components of the magnetic field B_{\parallel} and B_{\perp} appear weighted by the CLV law. In general, the components of the magnetic field may depend on the position on the stellar surface in a complicated manner, so that the previous integrals might not have closed expressions. In the simple case that the magnetic field is constant across the stellar surface, we recover $\langle B_{\parallel} \rangle = B_{\parallel}$, $\langle B_{\perp}^2 \cos 2\phi_B \rangle = B_{\perp}^2 \cos 2\phi_B$, and $\langle B_{\perp}^2 \sin 2\phi_B \rangle = B_{\perp}^2 \sin 2\phi_B$. One of the simplest non-trivial configuration we can consider explicitly is that of a dipolar field. The magnetic field vector at each surface point is then:

$$\mathbf{B}(\mathbf{r}) = -\frac{H_d}{2} [\mathbf{e} - 3(\mathbf{e} \cdot \mathbf{r})\mathbf{r}], \quad (9)$$

where the unitary vector \mathbf{e} defines the orientation of the dipole and the unit vector \mathbf{r} indicates positions on the stellar surface. The quantity H_d represents the magnetic field strength at the poles of the dipolar field. From Eq. 7 and Eq. 9 and after some algebra, it is possible to write the flux

of the Stokes parameters as:

$$\begin{aligned}\mathcal{F}_V(x) &= -\frac{1}{10}C\Lambda g \frac{15+u}{6-2u-3v} H_{\parallel} \frac{\partial \mathcal{F}_I(x)}{\partial x} \\ \mathcal{F}_Q(x) &= -\frac{C^2}{4480}\Lambda^2 G \frac{420-68u-105v}{6-2u-3v} H_{\perp}^2 \cos 2\phi_d \frac{\partial^2 \mathcal{F}_I(x)}{\partial x^2} \\ \mathcal{F}_U(x) &= -\frac{C^2}{4480}\Lambda^2 G \frac{420-68u-105v}{6-2u-3v} H_{\perp}^2 \sin 2\phi_d \frac{\partial^2 \mathcal{F}_I(x)}{\partial x^2},\end{aligned}\quad (10)$$

where $H_{\parallel} = H_d \cos \theta_d$ and $H_{\perp} = H_d \sin \theta_d$, θ_d being the inclination of the axis of the dipole while ϕ_d is the azimuth of the dipole (for a similar derivation see Landolfi et al. 1993).

The previous formalism has demonstrated that the weak-field approximation leads to formally the same expressions either if we consider specific intensities (that is applied whenever spatial resolution is available) or if we consider fluxes (whenever the object of interest cannot be resolved). In other words, the observed circular (linear) polarised spectrum is proportional to the first (second) derivative of the observed intensity through the longitudinal (orthogonal) component of the magnetic field. The only difference resides on the exact definition of the components of the field which, in the spatially unresolved case, are an average over the stellar surface weighted by the CLV. Therefore, for the sake of simplicity, it is possible to combine all expressions on the following general ones:

$$\begin{aligned}\mathcal{V} &= -C\mathcal{B}_{\parallel}\mathcal{I}' \\ \mathcal{Q} &= -C^2\mathcal{B}_{\perp}^2 \cos 2\phi\mathcal{I}'' \\ \mathcal{U} &= -C^2\mathcal{B}_{\perp}^2 \sin 2\phi\mathcal{I}''.\end{aligned}\quad (11)$$

The meaning of the newly defined variables is summarized in Tab. 1.

3 ESTIMATION OF THE MAGNETIC FIELD VECTOR

Once the model is set, our aim is to infer the magnetic field vector parametrised in terms of \mathcal{B}_{\parallel} , \mathcal{B}_{\perp} , and ϕ from

Table 1. Variables in Eq. 11.

Variable	Resolved	Dipole
\mathcal{I}	I	\mathcal{F}_I
\mathcal{V}	V	\mathcal{F}_V
\mathcal{Q}	Q	\mathcal{F}_Q
\mathcal{U}	U	\mathcal{F}_U
\mathcal{I}'	$\Lambda g \frac{\partial I(x)}{\partial x}$	$\frac{1}{10} \frac{15+u}{6-2u-3v} \Lambda g \frac{\partial \mathcal{F}_I(x)}{\partial x}$
\mathcal{I}''	$\frac{1}{4} \Lambda^2 G \frac{\partial^2 I(x)}{\partial x^2}$	$\frac{1}{4480} \frac{420-68u-105v}{6-2u-3v} \Lambda^2 G \frac{\partial^2 \mathcal{F}_I(x)}{\partial x^2}$
\mathcal{B}_{\parallel}	B_{\parallel}	H_{\parallel}
\mathcal{B}_{\perp}	B_{\perp}	H_{\perp}
ϕ	ϕ_B	ϕ_d
θ	θ_B	θ_d
\mathcal{B}	B	H_d

observations of a set of spectral lines ($\mathcal{I}^i, \mathcal{Q}^i, \mathcal{U}^i, \mathcal{V}^i$) with $i = 1 \dots n_{\text{lines}}$. Assuming that the weak field approximation can be applied for the set of observed spectral lines and that observations are corrupted with uncorrelated Gaussian noise, we can use a least-squares estimator (maximum likelihood) to retrieve the magnetic field vector (see the Appendix for more details). The χ^2 merit function is defined as:

$$\chi^2 = \sum_{ij} \frac{(\mathcal{V}_j^i - \mathcal{V}_j^{i;mod})^2}{(\sigma_{\mathcal{V}_j}^i)^2} + \sum_{ij} \frac{(\mathcal{Q}_j^i - \mathcal{Q}_j^{i;mod})^2}{(\sigma_{\mathcal{Q}_j}^i)^2} + \sum_{ij} \frac{(\mathcal{U}_j^i - \mathcal{U}_j^{i;mod})^2}{(\sigma_{\mathcal{U}_j}^i)^2}, \quad (12)$$

where the subindex j indicates the position along the coordinate x . The label *mod* refer to the model for Stokes profiles, and the superindex i labels the spectral lines. The previous merit functions consider the quite general case in which the standard deviation of the corrupting Gaussian noise is different for Stokes Q , U , and V and for each wavelength point x_j . However, in practice, we have the simpler situation in which $\sigma_{\mathcal{V}_j}^i \approx \sigma_{\mathcal{Q}_j}^i \approx \sigma_{\mathcal{U}_j}^i = \sigma_j^i$. Moreover, although the number of photons arriving in the line cores are smaller than in the far wings, we make the approximation that the noise variance is wavelength-independent. Furthermore, we also consider that it is independent of the considered line, so that $\sigma_j^i = \sigma$. These simplifications lead to less cluttered expressions for the inferred parameters. However, we point out that the general expressions that emerge from the optimization of Eq. (12) can be found in the Appendix.

In order to infer a certain parameter, we have to find the global minimum of the χ^2 . This is trivially obtained by solving the non-linear system of equations obtained by setting the derivatives of the χ^2 function with respect to that parameter to zero. By so doing (see Appendix), we can obtain the expression of the magnetic field vector in terms of the observables:

$$\begin{aligned} \mathcal{B}_{\parallel} &= -\frac{1}{C} \frac{\sum_{ij} \mathcal{V}_j^i \mathcal{I}_j^i}{\sum_{ij} (\mathcal{I}_j^i)^2} \\ \mathcal{B}_{\perp}^2 &= \frac{1}{C^2} \frac{\sqrt{(\sum_{ij} \mathcal{Q}_j^i \mathcal{I}_j^i)^2 + (\sum_{ij} \mathcal{U}_j^i \mathcal{I}_j^i)^2}}{\sum_{ij} (\mathcal{I}_j^i)^2} \\ \phi &= \frac{1}{2} \arctan \frac{\sum_{ij} \mathcal{U}_j^i \mathcal{I}_j^i}{\sum_{ij} \mathcal{Q}_j^i \mathcal{I}_j^i} + \phi_0, \end{aligned} \quad (13)$$

and the derived quantities:

$$\begin{aligned} \tan \theta &= \frac{\mathcal{B}_{\perp}}{\mathcal{B}_{\parallel}} \\ \mathcal{B} &= \sqrt{\mathcal{B}_{\parallel}^2 + \mathcal{B}_{\perp}^2}. \end{aligned} \quad (14)$$

The phase shift ϕ_0 is used to set the correct quadrant for the azimuth and depends on the sign of the numerator $\mathfrak{U} = \sum_{ij} \mathcal{U}_j^i \mathcal{I}_j^i$ and the denominator $\mathfrak{Q} = \sum_{ij} \mathcal{Q}_j^i \mathcal{I}_j^i$ as follows. If $\mathfrak{Q} \neq 0$ then:

$$\phi_0 = \begin{cases} 0 & \text{if } \mathfrak{U} \geq 0 \text{ and } \mathfrak{Q} > 0 \\ \pi & \text{if } \mathfrak{U} < 0 \text{ and } \mathfrak{Q} > 0 \\ \frac{\pi}{2} & \text{if } \mathfrak{Q} < 0 \end{cases} \quad (15)$$

In the case $\mathfrak{Q} = 0$ then:

$$\phi_0 = \begin{cases} \frac{\pi}{4} & \text{if } \mathfrak{U} > 0 \\ \frac{3\pi}{4} & \text{if } \mathfrak{U} < 0 \end{cases} \quad (16)$$

If both \mathfrak{Q} and \mathfrak{U} are zero, the angle ϕ is obviously undefined.

The estimated errors can be computed from the covariance matrix (e.g., Press et al. 1986). In the simple case we consider of equal wavelength-independent standard deviation for all spectral lines, the covariance matrix is diagonal (no correlation between the parameters), and the errors at a confidence level of 68.3% (one sigma) are expressed as:

$$\begin{aligned} \delta \mathcal{B}_{\parallel} &= \pm \frac{\sigma}{C \sqrt{\sum_{ij} (\mathcal{I}_j^i)^2}} \\ \delta \mathcal{B}_{\perp} &= \pm \frac{\sigma}{2C^2 \mathcal{B}_{\perp} \sqrt{\sum_{ij} (\mathcal{I}_j^i)^2}} \\ \delta \phi &= \pm \frac{\sigma}{2C^2 \mathcal{B}_{\perp}^2 \sqrt{\sum_{ij} (\mathcal{I}_j^i)^2}} \\ \delta \theta &= \pm \frac{\sqrt{\mathcal{B}_{\perp}^2 \delta \mathcal{B}_{\parallel}^2 + \mathcal{B}_{\parallel}^2 \delta \mathcal{B}_{\perp}^2}}{\mathcal{B}_{\parallel}^2 + \mathcal{B}_{\perp}^2} \\ \delta \mathcal{B} &= \pm \sqrt{\frac{\mathcal{B}_{\parallel}^2 \delta \mathcal{B}_{\parallel}^2 + \mathcal{B}_{\perp}^2 \delta \mathcal{B}_{\perp}^2}{\mathcal{B}_{\parallel}^2 + \mathcal{B}_{\perp}^2}}, \end{aligned} \quad (17)$$

4 BIAS OF THE MAXIMUM LIKELIHOOD ESTIMATOR

It is widely known that all maximum likelihood estimators may suffer from biases. The bias is the difference between the value of the estimator and the true value of the parameter. Each individual parameter has to be studied separately using analytical/numerical simulations to understand to which extent the estimation of \mathcal{B}_{\parallel} , \mathcal{B}_{\perp} , and ϕ are subjected to bias. For simplicity the simulations we present in the following refer to the resolved case in which we use specific intensities. However, the behaviour of the estimator is the same also for the stellar case.

In order to carry out the simulations, we focus on the Fe I line with central wavelength $\lambda_0 = 5250.2 \text{ \AA}$. This spectral line is produced by the transition $^5D_0 - ^7D_1$. The values of the effective Landé factor for circular polarisation is $g = 3$, while it is $G = 9$ for linear polarisation. We consider a constant magnetic field strength of $B = 500 \text{ G}$, which is sufficiently weak so that the weak-field approximation can still

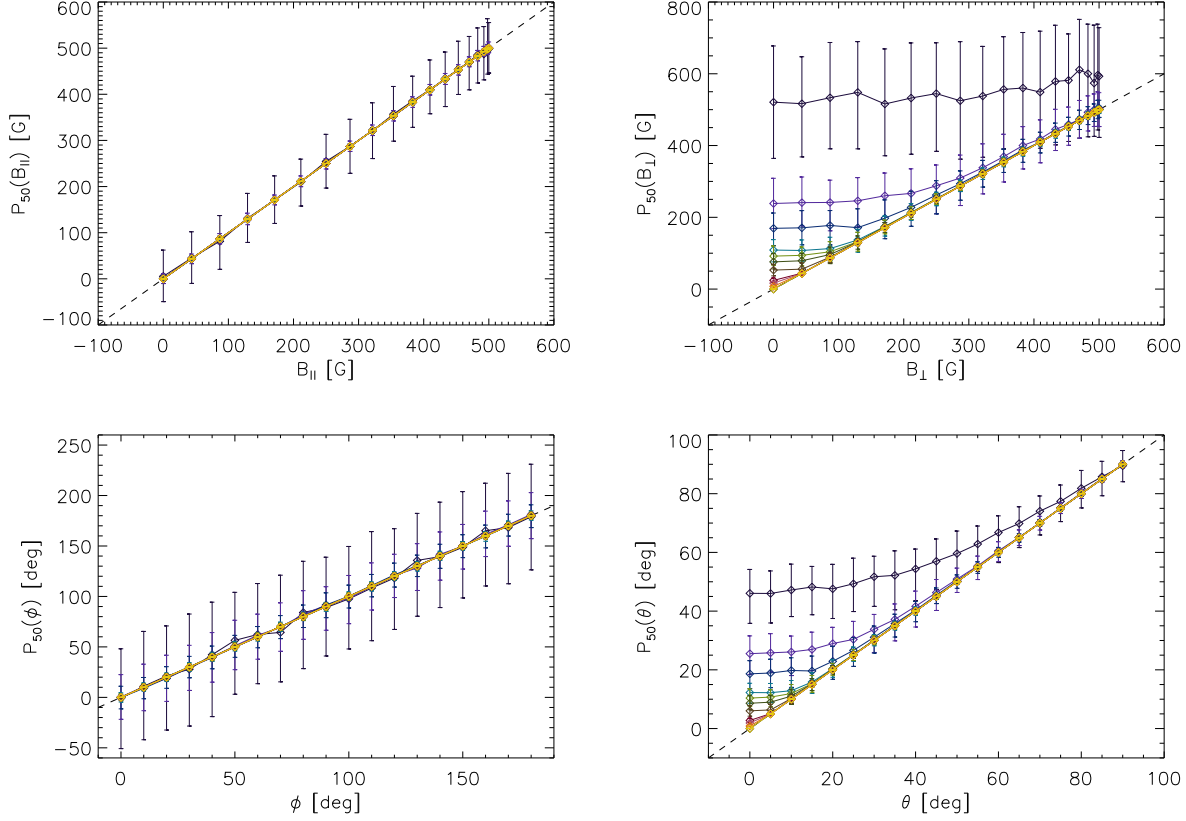


Figure 2. Bias of the maximum likelihood estimator of the magnetic field in the weak field approximation. Different color lines represent different values of the noise level added to the synthetic profiles of the Fe I line at 5250.2 Å. The color code is the following: yellow represents the inversion of the profiles without noise. Orange displays the results for a noise level of $10^{-5} I_c$, pink for $5 \times 10^{-5} I_c$, red for $10^{-4} I_c$, brown for $5 \times 10^{-4} I_c$, dark green for $10^{-3} I_c$, light green for $1.5 \times 10^{-3} I_c$, light blue for $2 \times 10^{-3} I_c$, dark blue for $5 \times 10^{-3} I_c$, light violet for $10^{-2} I_c$, and dark violet for $5 \times 10^{-2} I_c$. The I_c is the continuum intensity.

be considered. The inclination and the azimuth of the field are set to vary uniformly between 0 and 180° . Letting the azimuth vary in this interval, we avoid the 180° ambiguity in the azimuth present in the radiative transfer equation. For simplification, we consider a Gaussian intensity profile in the form:

$$I(\lambda) = 1 - d_c \exp \left[-\frac{(\lambda - \lambda_0)^2}{2w^2} \right], \quad (18)$$

where $d_c = 1/2$ and $w = 0.05 \text{ Å}$. The parameters for the Gaussian profile have been fixed to fit a solar observation obtained with the IMAx instrument. From the IMAx observational capabilities, it can be verified that the 5250.2 Å line is in the weak-field regime up to $\sim 1 \text{ kG}$ (with the line assumed to be in local thermodynamical equilibrium in a quiet Sun model atmosphere). We have synthesized 500 profiles for different combinations of the inclination and azimuth and different noise realisations. The noise added has a Gaussian statistics and we consider the effect of different standard deviations. We use Eqs. (13) to compute the inferred values of the parameters. Since we repeat the experiment for different realisations of the noise, we end up with a distribution of values for each parameter. We adopt the median value as the estimation of the parameter (the percentile 50 P_{50} , i. e., the value of the parameter that contains 50% of the area

of the distribution). To quantify the dispersion produced by noise we use the percentiles 16 and 84 (which encompass one standard deviation around the estimated value).

Figure 2 displays the inversion of the synthetic profiles with different values of the standard deviation of the noise, from 0 to 5×10^{-2} in units of the continuum intensity, I_c . It is clear from the upper and lower left panels that both the longitudinal component of the magnetic field and the azimuthal angle are unbiased quantities. This means that the estimated value statistically coincides with the original one. As expected, the dispersion grows with the increasing noise level.

Contrary to B_{\parallel} and ϕ_B (note that the notation for the magnetic field vector is the one associated to resolved sources), the transversal component of the magnetic field and hence the inclination angle presents a non-zero bias. Except for the case in which there is no added noise, small transversal components of the field (smaller or at the level of the noise amplitude) are overestimated. The fundamental reason is that the expression for B_{\perp} in Eq. (13) is not robust against noise. It can be verified that if Q_j^i and U_j^i are at the noise level (can be described as Gaussian distributions with zero mean and variance σ^2), B_{\perp} is a random variable

following the probability distribution:

$$p(B_{\perp}) = \frac{2C^4 \left(\sum_{ij} (\mathcal{I}''^i_j)^2 \right)}{\sigma^2} B_{\perp}^3 \exp \left[-\frac{C^4 \sum_{ij} (\mathcal{I}''^i_j)^2 B_{\perp}^4}{2\sigma^2} \right]. \quad (19)$$

The value of this distribution for a given percentile c fulfils:

$$B_{\perp}^c = \left(\frac{-2 \ln(1-c)}{\sum_{ij} (\mathcal{I}''^i_j)^2} \right)^{1/4} \frac{\sqrt{\sigma}}{C}. \quad (20)$$

The percentile 50 ($c = 0.5$) correctly captures the value of the bias at small values of B_{\perp} . Once this value is computed, if the inferred value of B_{\perp} is similar to this value, one should be aware that the correct value of B_{\perp} might be smaller. If this is not taken into account, the estimated inclination of the field is larger than the real one and artificially horizontal fields might be inferred. Note also that in the stellar case, the inferred inclination of the dipole axis would be more inclined than the real one.

5 THE PARTICULAR CASE OF THE LEAST-SQUARES-DECONVOLUTION PROFILE

Most detections of faint signals in stellar atmospheres have been evidenced by adding many spectral lines. The polarimetric signal per spectral resolution element is known to be well below the noise level, so that line addition is a must to fight against photon noise. The most widely used and successful technique that combines the information of many spectral lines is the Least-Squares Deconvolution (LSD) technique (Donati et al. 1997). The equations presented in this paper allow us to retrieve the magnetic field vector from the polarised stellar spectra taking into account many spectral lines. Thanks to it, we can rewrite the equations to directly extract information from the LSD profiles.

The LSD technique is fundamentally based on the application of the weak-field approximation and on the assumption of a constant CLV for all spectral lines (in our case, u and v are set constant). This means that all Stokes profiles for each spectral line can be computed with a single spectral profile whose proportionality constant changes from line to line (see Donati et al. 1997):

$$\mathcal{I}^i = \eta^i \bar{\mathcal{I}} \quad (21)$$

$$\mathcal{V}^i = \eta^i \Lambda^i g^i \bar{\mathcal{V}} \quad (22)$$

$$\mathcal{Q}^i = \eta^i (\Lambda^i)^2 G^i \bar{\mathcal{Q}} \quad (23)$$

$$\mathcal{U}^i = \eta^i (\Lambda^i)^2 G^i \bar{\mathcal{U}}, \quad (24)$$

where η^i is the line depth and $\bar{\mathcal{I}}$, $\bar{\mathcal{Q}}$, $\bar{\mathcal{U}}$, and $\bar{\mathcal{V}}$ are the LSD profiles that can be computed using a least-squares procedure as explained in Donati et al. (1997). **We refer to Kochukhov, Makaganiuk & Piskunov (2010) for an in-depth analysis of the assumptions and potential problems of LSD.** From the previous equations, it is easy to show that it is possible to infer the magnetic field vector

using:

$$\begin{aligned} \mathcal{B}_{\parallel} &= -\frac{1}{CK} \frac{\sum_j \bar{\mathcal{V}}_j \frac{\partial \bar{\mathcal{I}}_j}{\partial x}}{\sum_j \left(\frac{\partial \bar{\mathcal{I}}_j}{\partial x} \right)^2} \\ \mathcal{B}_{\perp} &= \frac{1}{C^2 K'} \frac{\sqrt{\left(\sum_j \bar{\mathcal{Q}}_j \frac{\partial^2 \bar{\mathcal{I}}_j}{\partial x^2} \right)^2 + \left(\sum_j \bar{\mathcal{U}}_j \frac{\partial^2 \bar{\mathcal{I}}_j}{\partial x^2} \right)^2}}{\sum_j \left(\frac{\partial^2 \bar{\mathcal{I}}_j}{\partial x^2} \right)^2} \\ \phi &= \frac{1}{2} \arctan \frac{\sum_j \bar{\mathcal{U}}_j \frac{\partial^2 \bar{\mathcal{I}}_j}{\partial x^2}}{\sum_j \bar{\mathcal{Q}}_j \frac{\partial^2 \bar{\mathcal{I}}_j}{\partial x^2}} + \phi_0, \end{aligned} \quad (25)$$

where $K = 1$ and $K' = 1/4$ for the case of an unresolved star with constant magnetic field and

$$K = \frac{1}{10} \frac{15 + u}{6 - 2u - 3v}, \quad K' = \frac{1}{4480} \frac{420 - 68u - 105v}{6 - 2u - 3v} \quad (26)$$

for the stellar dipole. Note that it is not necessary to assume averaged atomic parameters for the LSD profile, treating it as a *mean* spectral line. In our case, the estimated magnetic field vector only depends on the observables, the atomic parameters of each observed spectral line (which is necessary to compute the LSD profile), and the assumed CLV coefficients.

6 ILLUSTRATIVE EXAMPLES

The inference power of the expressions developed in the previous sections are illustrated with the aid of two different examples. The first one consists of a simulated stellar dipole using a Milne-Eddington (e.g., Landi Degl'Innocenti & Landolfi 2004) atmosphere and the second one is a particularly interesting observational example in which we can illustrate the effect of the bias in the transverse component of the magnetic field with spatial resolution.

Considering the stellar dipole, we simulate a Milne-Eddington atmosphere with a source function that varies as:

$$S(\tau) = S_0(1 + \beta\tau). \quad (27)$$

For simplicity, we choose $\beta = 1$ and consider a static atmosphere at all points of the stellar surface. We assume a spectral line centred at $\lambda = 5000 \text{ \AA}$ with a Doppler broadening of 0.04 \AA . Both the circular and linear effective Landé factors are equal to 1. The CLV variation of the Milne-Eddington atmosphere (that we have forced to be wavelength independent) gives $u = 2/3$. We integrate the Stokes signals on the visible stellar surface and we add Gaussian noise to the final integrated flux with a standard deviation of 5×10^{-5} in units of the intensity flux at the continuum, \mathcal{F}_c .

Figure 3 shows the flux of the simulated Stokes parameters coming from a dipole field with the following parameters: $H_d = 1500 \text{ G}$, $\theta_d = 80^\circ$, and $\phi_d = 25^\circ$. The black lines represent the synthetic fluxes, and the rhombs the synthetic observations with added noise. We invert the noisy Stokes fluxes using Eqs. 13, 14 and 17 and obtain $H_{\parallel} = 265.5 \pm 2.4 \text{ G}$, $H_{\perp} = 1501.6 \pm 25.7 \text{ G}$, $\phi_d = 27.2^\circ \pm 2.0$, and the derived quantities $H_d = 1524.9 \pm 25.3 \text{ G}$ and $\theta_d = 80.0^\circ \pm 0.2$. All quantities are nicely recovered. In fact, the bias estimations

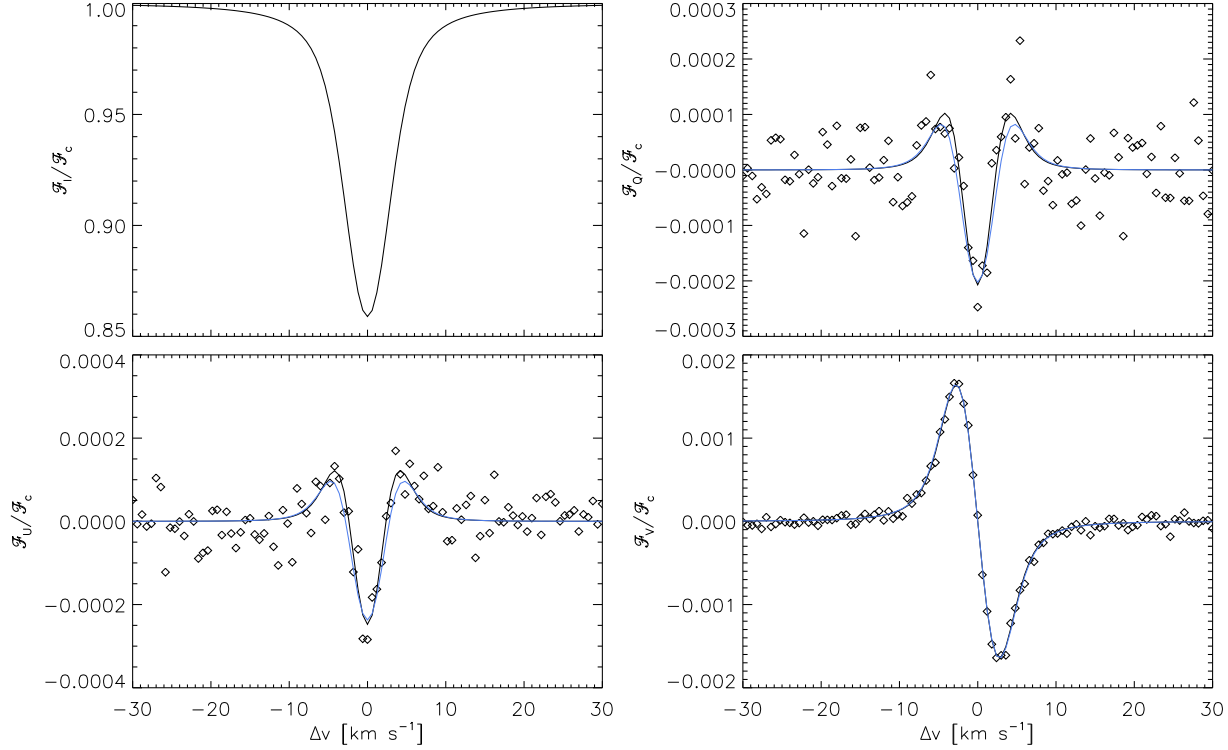


Figure 3. Inversion of a synthetic stellar dipole. The black lines display the synthetic flux of a dipole with a strength $H_d=1500$ G, an inclination $\theta_d = 80^\circ$ with respect to the line of sight and an azimuth of $\theta_d = 25^\circ$. The rhombs are the synthetic polarised flux including an additive Gaussian noise with a standard deviation of $5 \times 10^{-5} \mathcal{F}_c$. The blue lines are the fluxes inferred from the inversion using Eq. 13.

for the perpendicular component H_\perp for the percentiles 16, 50, and 84 (i. e. containing one sigma probability) are 302.0, 426.5, and 543.8 G, respectively. This means that the computed H_\perp is reliable. Now, we assume a weak dipole with $H_d = 100$ G, $\theta_d = 20^\circ$, and $\phi_d = 25^\circ$, and the same noise level. The inferred parameters are $H_\parallel = 99.7 \pm 2.3$ G, $H_\perp = 333.9 \pm 109.0$ G, $\phi_d = 39.8^\circ \pm 37.4$, $H_d = 348.3 \pm 104.5$ G, and $\theta_d = 73.4^\circ \pm 5.1$. In principle, both the longitudinal and the transverse components (as well as the inclination) should be well recovered but the azimuth remains undetermined. However, the bias of H_\perp for the percentiles 16, 50, and 84 are 293.2, 414.1, and 528.0 G, respectively. Therefore, the inferred perpendicular component of the dipole is consistent with a bias. This implies that the inferred value, although having a small (Gaussian) error, it has to be considered an upper limit. In this case, we know that the perpendicular component of the dipole is very small (34 G). Consequently, we overestimate both the perpendicular component and the intrinsic strength of the dipole. Additionally, the dipole appears to be much more inclined than in reality.

The second experiment consists of filter-polarimetric data observed with the IMaX instrument on the Sunrise mission. This polarimeter observes the 5250.2 FeI line (for which we have carried out the bias experiment in Section 4). The data set consists of the four Stokes parameters observed at the quietest areas of the solar disc centre at a spatial cutoff frequency of about 0.15-0.18'' (~ 120 km in the solar surface, the best one at the moment for instruments with polarimetric capabilities). The noise level in circular

and linear polarization is 10^{-3} in units of the continuum intensity, I_c . The left panel of Fig. 4 shows the continuum intensity image, with brighter areas associated to granular regions, where the plasma ascends to the photosphere. Dark areas are the intergranular lanes, where the motions are preferentially downflowing. The right panel of this same figure displays the estimated bias for the transversal component of the magnetic field using Eq. 20 for the percentile $c = 0.5$. The bias has a very particular spatial distribution which mimics reversed granulation (bright areas become dark and viceversa). This particular spectral line is strongly sensitive to the temperature, becoming very deep and narrow in intergranular lanes and less deep and broad in granules. The dependence of Eq. 20 on the second derivative of the intensity profile, which gives an idea of the width of the spectral line, is the reason why the bias is larger in intergranules and less important in granules.

Left panel of Fig. 5 displays the inferred longitudinal component of the magnetic field. As can be seen, the noisy background has values around zero, consistent with the fact that the estimator of this quantity is unbiased. The central panel shows the inferred transversal component of the magnetic field. The noisy background is now filled by magnetic fields that have quite intense values, illustrating the network pattern of the bias. If this bias for the longitudinal field is not appropriately account for, this might lead to an artificial excess of inclined magnetic fields. This effect could be affecting some of the recent magnetic field inferences in the quiet Sun observed with Hinode (see,

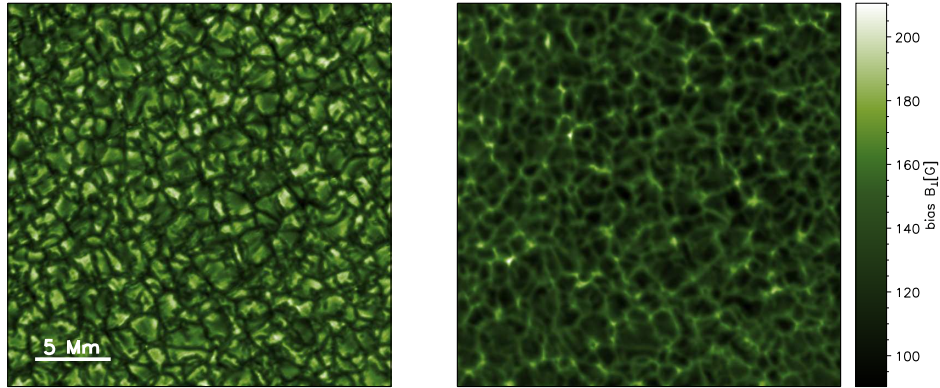


Figure 4. Left: continuum intensity at 227 mÅ from the core of the 5250.2 Å. Right: Median value of the bias for the transverse component of the magnetic field computed with Eq. 20 and $c = 0.5$.

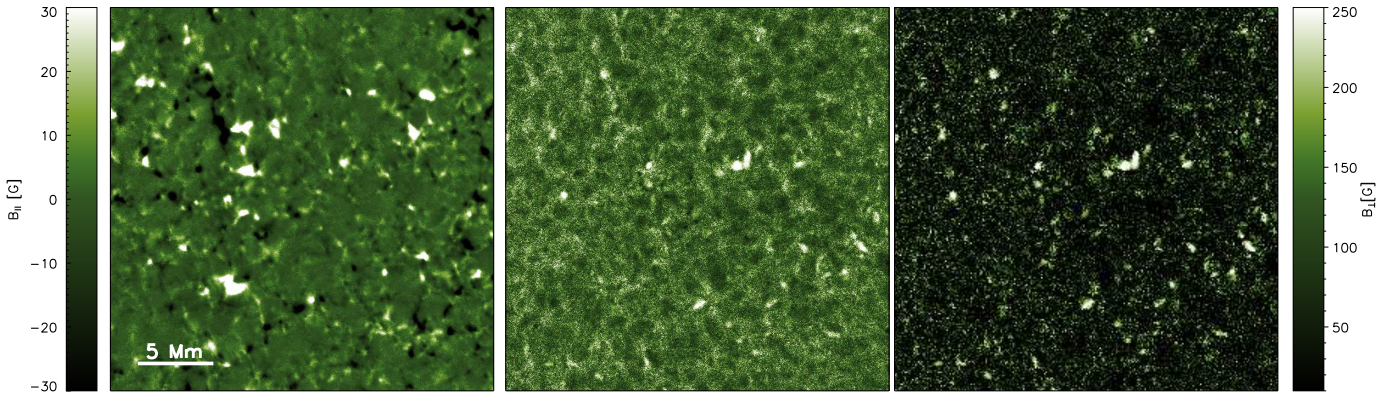


Figure 5. Left (centre): inferred value of the longitudinal (transverse) magnetic field for the 5250.2 Å line. Right: transverse field corrected from the bias due to pure noise. Central and right panels share the same colorbar.

e.g., Orozco Suárez et al. 2007; Lites et al. 2008; Sheminova 2009; Ishikawa & Tsuneta 2009, 2010, 2011).

Since we can characterize the bias by its median value, it is possible to distinguish real signals from false signals produced by the presence of noise. The way we proceed is as follows. We compute a conservative upper limit for a trustful field as the bias for the percentile 84 ($c = 0.84$ in Eq. 20). This value changes from pixel to pixel. Then, we force $B_{\perp} = 0$ in those places where the inferred B_{\perp} is smaller than the bias. The result is represented in the right panel of Fig. 5. Now, the real signals (coming from pixels with linear polarization clearly above the noise level) are much more evident and most of the background has disappeared. Note that we have only removed those signals that were produced by the presence of noise (the expected value of $B_{\perp} = 0$ should be zero). However, if $B_{\perp} \neq 0$, the bias can still be important if the noise level is high, as shown in Fig. 2.

7 CONCLUSIONS

We have shown that the weak field approximation (the Stokes parameters are proportional to the first and second order derivative of the intensity) also holds for observed stellar fluxes in the case of slow rotators. We have used a maximum likelihood estimator to infer the magnetic field vector from the observed Stokes profiles. The main result of this

papers is that we give explicit formulae for the components of the magnetic field vector in terms of the observables. The formulae are general and hold for specific intensities, for integrated fluxes and are slightly modified for LSD profiles. In the particular case of a stellar dipole, the orientation of the dipole axis and its intensity can be recovered from one observation of the full Stokes vector.

We have also studied the bias of this maximum likelihood estimator. The longitudinal magnetic field and the azimuthal angle are unbiased quantities. However, the transversal component of the magnetic field and hence the inclination of the field are overestimated in the presence of noise. We derive the estimated value of the perpendicular component of the magnetic field in the case that there is no linear polarization signal above the noise (the bias for $B_{\perp}=0$). We propose to evaluate this bias prior to the inference of the perpendicular component of the field. One should be very cautious when the inferred B_{\perp} is of the order of the bias.

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APPENDIX A: DERIVATION OF THE MAXIMUM LIKELIHOOD SOLUTION OF THE STELLAR MAGNETIC FIELD

We start from the general weak-field equations that hold both for the resolved (solar) case and for the integrated stellar dipole (Eq. 11):

$$\begin{aligned}\mathcal{V} &= -C\mathcal{B}_{\parallel}\mathcal{I}' \\ \mathcal{Q} &= -C^2\mathcal{B}_{\perp}^2 \cos 2\phi\mathcal{I}'' \\ \mathcal{U} &= -C^2\mathcal{B}_{\perp}^2 \sin 2\phi\mathcal{I}'',\end{aligned}\tag{A1}$$

where the explicit expressions for $\mathcal{V}, \mathcal{Q}, \mathcal{U}, \mathcal{I}'$, and \mathcal{I}'' , for each case can be found in Table 1 in the main text. We denote the Stokes vector by $\mathcal{S} = [\mathcal{I}, \mathcal{Q}, \mathcal{U}, \mathcal{V}]$. Assuming that the difference between the data and the model follows a Gaussian distribution, the likelihood (the probability distribution of the data given the parameters) can be written as:

$$\mathcal{L} = \prod_{i=1}^{n_l} \prod_{j=1}^{n_{\lambda}} \prod_{k=2}^4 \left[2\pi(\sigma_{jk}^i)^2 \right]^{-1/2} \exp \left[-\frac{(\mathcal{S}_{jk}^i - \mathcal{S}_{jk}^{i;\text{mod}})^2}{2(\sigma_{jk}^i)^2} \right],\tag{A2}$$

where the index i refers to the spectral line, j refers to the wavelength, and k to the element of the Stokes vector. The abbreviation “mod” stands for the model. The symbol σ^2 stands for the variance of $\mathcal{S} - \mathcal{S}^{\text{mod}}$. Taking the logarithm, we build the log-likelihood, $\ln \mathcal{L}$:

$$\ln \mathcal{L} = -\frac{3n_l n_{\lambda}}{2} \ln(2\pi) - \sum_{i=1}^{n_l} \sum_{j=1}^{n_{\lambda}} \sum_{k=2}^4 \left[\frac{(\mathcal{S}_{jk}^i - \mathcal{S}_{jk}^{i;\text{mod}})^2}{2(\sigma_{jk}^i)^2} + \frac{1}{2} \ln(\sigma_{jk}^i)^2 \right].\tag{A3}$$

In order to estimate the parameters that fit the data given the proposed model we have to maximise the likelihood or, equivalently, minimise $\ln \mathcal{L}$. Let $\mathbf{p} = [\mathcal{B}_{\parallel}, \mathcal{B}_{\perp}, \phi]$ denote the set of parameters of our model. Following the standard approach, to estimate the vector of parameters \mathbf{p} , we must solve the following set of equations:

$$\frac{\partial \ln \mathcal{L}}{\partial p_l} = 0 = \sum_{i=1}^{n_l} \sum_{j=1}^{n_{\lambda}} \sum_{k=2}^4 \frac{\partial}{\partial p_l} \left[\frac{(\mathcal{S}_{jk}^i - \mathcal{S}_{jk}^{i;\text{mod}})^2}{(\sigma_{jk}^i)^2} + \ln(\sigma_{jk}^i)^2 \right], \quad \forall l.\tag{A4}$$

There is an important point to clarify given that, in our case, the model is not fully analytical because it depends on the observed intensity profile (Asensio Ramos & Manso Sainz 2011). Therefore, assuming no correlation between the model and the observable (the only difference being produced by uncorrelated Gaussian noise), the variance for each Stokes parameter is given by:

$$(\sigma_2)^2 = \sigma^2(\mathcal{S}_2) + \sigma^2(\mathcal{S}_2^{\text{mod}}) = \sigma^2(\mathcal{S}_2) + C^4 \mathcal{B}_{\perp}^4 \cos^2 2\phi \sigma^2(\mathcal{I}'')\tag{A5}$$

$$(\sigma_3)^2 = \sigma^2(\mathcal{S}_3) + \sigma^2(\mathcal{S}_3^{\text{mod}}) = \sigma^2(\mathcal{S}_3) + C^4 \mathcal{B}_{\perp}^4 \sin^2 2\phi \sigma^2(\mathcal{I}'')\tag{A6}$$

$$(\sigma_4)^2 = \sigma^2(\mathcal{S}_4) + \sigma^2(\mathcal{S}_4^{\text{mod}}) = \sigma^2(\mathcal{S}_4) + C^2 \mathcal{B}_{\parallel}^2 \sigma^2(\mathcal{I}').\tag{A7}$$

Thanks to the previous equations, the variances that appear in Eq. (A4) depend on the actual parameters, which makes the minimisation much harder. Luckily, in the weak field regime and for the observational spectral resolution of interest nowadays, it is easy to verify that $\sigma^2(\mathcal{S}_i^{\text{mod}}) \leq \sigma^2(\mathcal{S}_i)$. In any case, this condition should be checked before carrying out any inversion using the formulae derived in this work. Assuming a first order approximation to the derivative \mathcal{I}' , we find:

$$\sigma^2(\mathcal{I}') = 2K^2 \frac{\sigma^2(\mathcal{S}_1)}{\Delta x^2},\tag{A8}$$

where $K = 1$ for the resolved case and $K = \frac{1}{10} \frac{15+u}{6-2u-3v}$ for the dipole case. Assuming $\sigma^2(\mathcal{S}_i^{\text{mod}}) \leq \sigma^2(\mathcal{S}_i)$ and that the noise in intensity is the same as in the circular polarisation, it holds that:

$$\Delta x \geq \sqrt{2} C \mathcal{B}_{\parallel} K \Lambda g,\tag{A9}$$

which means that the spectral sampling has to be larger than or of the order of the Zeeman splitting. For instance, for a wavelength of 5000 Å, a Landé factor of 1.5, and $B_{\parallel} = 500$ G, the spectral sampling has to be larger or equal to 12 mÅ for $K = 1$. Luckily, this is the case in most of the observational cases. For example, for the same wavelength, the expected sampling for two spectrographs with a resolving power of $R = 60000, 300000$ are 83 mÅ and 16 mÅ, respectively. For the linear polarisation we have, if we assume that their associated noise equals the noise in intensity, that:

$$\begin{aligned}\Delta x &\geq \sqrt{2GK' \cos 2\phi} C \mathcal{B}_{\perp} \Lambda \\ \Delta x &\geq \sqrt{2GK' \sin 2\phi} C \mathcal{B}_{\perp} \Lambda,\end{aligned}\tag{A10}$$

where $K' = 1/4$ for the resolved case and $K' = \frac{1}{4480} \frac{420-68u-105v}{6-2u-3v}$ for the dipole case. For $B_{\perp} = 500$ G, $\phi = 0^\circ$ and $G = 2.25$, we end up with $\Delta x = 6$ mÅ.

Taking the previous considerations into account, Eq. (A4) can be simplified to:

$$\frac{\partial \ln \mathcal{L}}{\partial p_l} = 0 = \sum_{i=1}^{n_l} \sum_{j=1}^{n_\lambda} \sum_{k=2}^4 \frac{\partial}{\partial p_l} \left[\frac{(\mathcal{S}_{jk}^i - \mathcal{S}_{jk}^{i;\text{mod}})^2}{(\sigma_{jk}^i)^2} \right] \equiv \frac{\partial \chi^2}{\partial p_l}, \quad (\text{A11})$$

where the well known χ^2 merit function is defined as:

$$\chi^2 = \sum_{ij} \frac{(\mathcal{V}_j^i - \mathcal{V}_j^{i;\text{mod}})^2}{(\sigma_{\mathcal{V}_j}^i)^2} + \sum_{ij} \frac{(\mathcal{Q}_j^i - \mathcal{Q}_j^{i;\text{mod}})^2}{(\sigma_{\mathcal{Q}_j}^i)^2} + \sum_{ij} \frac{(\mathcal{U}_j^i - \mathcal{U}_j^{i;\text{mod}})^2}{(\sigma_{\mathcal{U}_j}^i)^2}. \quad (\text{A12})$$

Explicitly, the derivatives of the χ^2 with respect to the parameters we want to infer are:

$$\frac{\partial \chi^2}{\partial \mathcal{B}_\parallel} = 2C \sum_{ij} \frac{\mathcal{V}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{V}_j}^i)^2} + 2C^2 \mathcal{B}_\parallel \sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{V}_j}^i)^2} \quad (\text{A13})$$

$$\frac{\partial \chi^2}{\partial \mathcal{B}_\perp} = 4C^2 \mathcal{B}_\perp \left(\cos 2\phi \sum_{ij} \frac{\mathcal{Q}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{Q}_j}^i)^2} + \sin 2\phi \sum_{ij} \frac{\mathcal{U}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{U}_j}^i)^2} \right) + 4C^4 \mathcal{B}_\perp^3 \left(\cos^2 2\phi \sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{Q}_j}^i)^2} + \sin^2 2\phi \sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{U}_j}^i)^2} \right) \quad (\text{A14})$$

$$\frac{\partial \chi^2}{\partial \phi} = 4C^2 \mathcal{B}_\perp^2 \left(\cos 2\phi \sum_{ij} \frac{\mathcal{U}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{U}_j}^i)^2} - \sin 2\phi \sum_{ij} \frac{\mathcal{Q}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{Q}_j}^i)^2} \right) + 4C^4 \mathcal{B}_\perp^4 \sin 2\phi \cos 2\phi \left(\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{U}_j}^i)^2} - \sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{Q}_j}^i)^2} \right) \quad (\text{A15})$$

By forcing these derivatives to zero, we obtain the maximum-likelihood estimate of each parameter. For the longitudinal magnetic field, using Eq. A13, we obtain a unique solution:

$$\mathcal{B}_\parallel = -\frac{1}{C} \frac{\sum_{ij} \frac{\mathcal{V}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{V}_j}^i)^2}}{\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{V}_j}^i)^2}}. \quad (\text{A16})$$

Dividing Eq. A14 and A15, we obtain a solution for the azimuth:

$$\tan 2\phi = \frac{\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{Q}_j}^i)^2} \sum_{ij} \frac{\mathcal{U}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{U}_j}^i)^2}}{\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{U}_j}^i)^2} \sum_{ij} \frac{\mathcal{Q}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{Q}_j}^i)^2}}. \quad (\text{A17})$$

Dividing Eq. A15 by $\cos 2\phi$ and using Eq. A17, after some algebra we obtain two solutions. One is $\mathcal{B}_\perp = 0$, which is not valid since this solution maximises the χ^2 . We have tested this computing the second derivative. The solution that minimises the χ^2 is given by:

$$\mathcal{B}_\perp^2 = \frac{1}{C^2} \frac{\sqrt{\left(\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{U}_j}^i)^2} \sum_{ij} \frac{\mathcal{Q}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{Q}_j}^i)^2} \right)^2 + \left(\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{Q}_j}^i)^2} \sum_{ij} \frac{\mathcal{U}_j^i \mathcal{I}_j^{'i}}{(\sigma_{\mathcal{U}_j}^i)^2} \right)^2}}{\sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{U}_j}^i)^2} \sum_{ij} \frac{(\mathcal{I}_j^{'i})^2}{(\sigma_{\mathcal{Q}_j}^i)^2}}. \quad (\text{A18})$$

The errors associated with each parameter are computed assuming that the surface around the maximum likelihood is approximately a multidimensional Gaussian. This is equivalent to assuming a parabolic approximation to the χ^2 close to the minimum. The curvature close to the minimum is given by the Hessian matrix ζ , whose elements are:

$$\zeta_{kl} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial p_k \partial p_l}. \quad (\text{A19})$$

The matrix can be calculated using the following second derivatives:

$$\frac{\partial^2 \chi^2}{\partial \mathcal{B}_{\parallel}^2} = 2C^2 \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{V}j}^i)^2} \quad (\text{A20})$$

$$\frac{\partial^2 \chi^2}{\partial \mathcal{B}_{\perp}^2} = 8C^4 \mathcal{B}_{\perp}^2 \left(\cos^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} + \sin^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} \right) \quad (\text{A21})$$

$$\frac{\partial^2 \chi^2}{\partial \phi^2} = 8C^4 \mathcal{B}_{\perp}^4 \left(\sin^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} + \cos^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} \right) \quad (\text{A22})$$

$$\frac{\partial \chi^2}{\partial \mathcal{B}_{\parallel} \partial \mathcal{B}_{\perp}} = \frac{\partial \chi^2}{\partial \mathcal{B}_{\perp} \partial \mathcal{B}_{\parallel}} = 0 \quad (\text{A23})$$

$$\frac{\partial \chi^2}{\partial \mathcal{B}_{\parallel} \partial \phi} = \frac{\partial \chi^2}{\partial \phi \partial \mathcal{B}_{\parallel}} = 0 \quad (\text{A24})$$

$$\frac{\partial \chi^2}{\partial \mathcal{B}_{\perp} \partial \phi} = \frac{\partial \chi^2}{\partial \phi \partial \mathcal{B}_{\perp}} = 8C^4 \mathcal{B}_{\perp}^3 \sin 2\phi \cos 2\phi \left(\sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} - \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} \right). \quad (\text{A25})$$

The square root of the diagonal of the covariance matrix gives the error estimates for each parameter. Such covariance matrix is just the inverse of the Hessian matrix, $\mathcal{C} = \zeta^{-1}$. In our case:

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_{11} & 0 & 0 \\ 0 & \mathcal{C}_{22} & \mathcal{C}_{23} \\ 0 & \mathcal{C}_{32} & \mathcal{C}_{33} \end{bmatrix}, \quad (\text{A26})$$

with

$$\mathcal{C}_{11} = \left[C^2 \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{V}j}^i)^2} \right]^{-1} \quad (\text{A27})$$

$$\mathcal{C}_{22} = \frac{1}{4C^4 \mathcal{B}_{\perp}^2} \frac{\sin^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} + \cos^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2}}{\cos^2 2\phi \sin^2 2\phi \left(\sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} - \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} \right)^2 + \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2}} \quad (\text{A28})$$

$$\mathcal{C}_{23} = \mathcal{C}_{32} = \frac{1}{4C^4 \mathcal{B}_{\perp}^3} \frac{\sin 2\phi \cos 2\phi \left(\sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} - \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} \right)^2}{\cos^2 2\phi \sin^2 2\phi \left(\sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} - \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} \right)^2 + \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2}} \quad (\text{A29})$$

$$\mathcal{C}_{33} = \frac{1}{4C^4 \mathcal{B}_{\perp}^4} \frac{\cos^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} + \sin^2 2\phi \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2}}{\cos^2 2\phi \sin^2 2\phi \left(\sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} - \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2} \right)^2 + \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{Q}j}^i)^2} \sum_{ij} \frac{(\mathcal{I}''_j)^2}{(\sigma_{\mathcal{U}j}^i)^2}} \quad (\text{A30})$$

The error bars are thus given by:

$$\delta p_k = \pm \Delta \sqrt{\mathcal{C}_{kk}}, \quad (\text{A31})$$

with $\Delta = 1, 1.65, 2, 2.57, 3$, and 3.89 for a confidence level of 68.3%, 90%, 95.4%, 99%, 99.73%, and 99.99%, respectively. This computation assumes that, to estimate the error of a parameter, we fix the values of the rest to the ones that maximize the likelihood and compute the confidence levels for the one-dimensional probability distribution of this parameter (see Press et al. 1986, for more details). Note that this approximation does not take into account the degeneracies between parameters since it does not integrate the probability distribution of the rest of the parameters but fixes a certain value (see Asensio Ramos 2011, for a robust Bayesian inversion). Note also that the covariance matrix is diagonal if the standard deviation of \mathcal{Q} and \mathcal{U} are the same ($\sigma_{\mathcal{Q}j}^i = \sigma_{\mathcal{U}j}^i$). This is generally the case in solar observations in which the measurement efficiencies for linear polarization are similar.